Characterization of the Distribution With Uniform Measure I(p): Continuous and Discrete Case

Caratterizzazione della distribuzione con misura I(p) uniforme: caso continuo e discreto

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1. The uniform inequality measure I(p)

A new inequality point measure I(p) has been recently introduced, based on the ratio between the lower mean and the upper mean (Zenga, 2007). This note is dedicated to a particular problem of characterization of the distribution of a variable X, using the point inequality measure I(p). In particular we are interested in obtaining the distribution function $F(x)$ of a non-negative random variable (r.v.) $X$ with uniform point measure I(p), for $0 < p < 1$. The interest toward this problem arises from the consideration that, being unlikely to achieve a real situation with perfect equality, it may be more realistic to try approaching a condition with uniform local inequality. In this way the inequality between the lower group, with values of $X$ less than or equal to $x$, and the upper group, with values of $X$ greater than $x$, is always the same for every value $x$ of the variable $X$.

2. The random variable with uniform measure I(p)

Let $X$ be an absolute continuous r.v. with distribution function $F(x)$ and probability density function $f(x)$, such that $\int_a^b f(x)dx = 1$, where $0 \leq a < b \leq +\infty$, and with finite and positive expected value $\mu$. The point inequality measure $I(p)$ is defined by

$$I(p) = 1 - \frac{M^-(p)}{M^+(p)} \quad 0 < p < 1; \quad \text{where} \quad M^-(p) = \frac{1}{p} \int_0^p x(t)dt, \quad M^+(p) = \frac{1}{1-p} \int_p^1 x(t)dt.$$  \hspace{1cm} (1)

In other words, $M^+(p)$ and $M^-(p)$ are respectively the lower mean and the upper mean referred to $p=F(x)$, with $x=a$, being $x(p) = F^{-1}(p) = \inf\{x:F(x) \geq p\}$.

By assuming the condition

$$I(p) = 1 - k \quad 0 < p < 1$$

where $k$, for $0 < k < 1$, is the level of uniformity, it is possible to obtain the expression of the distribution function $F(x)$ of the r.v. $X$, with finite and positive expected value $\mu$. By
the condition (1) and by remembering that \( \int_0^p x(t) \, dt = \mu - \int_p^1 x(t) \, dt \), we get \( x(p) = F^{-1}(p) = \mu k / [1 - p(1 - k)]^2 \). By inverting this last relation and the consideration that \( \lim_{x \to a} F(x) = 0 \), and \( \lim_{x \to b} F(x) = 1 \), for \( 0 \leq a < b \leq +\infty \), the distribution function results in

\[
F(x) = \begin{cases} 
0 & x \leq \mu k \\
(1 - k)^{-1} \left[ 1 - (\mu k)^{1/2} x^{-1/2} \right] & \mu k < x < \mu / k \\
1 & x \geq \mu / k
\end{cases}
\]

with \( 0 < k < 1 \) and \( \mu > 0 \).

It is easy to show that this r.v. is equal to a truncated Pareto distribution with the lower limit \( \mu k \), the upper limit \( \mu / k \), and the other parameter equal to 1/2 (cfr. Johnson, et al., 1994, p. 608).

Assume now that \( X \) is a discrete r.v. taking on \( N \) different values: \( 0 \leq a \leq x_1 < \ldots < x_j < \ldots < x_N \) \( \leq b < +\infty \), every one with probability \( 1/N \) and such that \( T = \sum_{j=1}^N x_j \). Let \( p_j = j / N \) be the share of values of \( X \) less than or equal to \( x_j \), the point inequality measure between the lower group and the upper group related to \( p_j \) is

\[
I(p_j) = 1 - M^-(p_j) \quad j = 1, \ldots, N;
\]

where \( M^-(p_j) = \sum_{i=1}^j x_i / j \), \( M^+(p_j) = \left\{ \left( T - \sum_{i=1}^j x_i \right) / (N - j) \right\} \) for \( j = 1, \ldots, N - 1 \) and \( x_{N+1}^* = T / (Nk) \); hence, by successive differences: \( \sum_{i=1}^j x_i - \sum_{i=1}^{j-1} x_i \) for \( j = 2, \ldots, N \), it is possible to get \( x_j = kTN [N - j(1 - k)]^{-1} [N - (j - 1)(1 - k)]^{-1} \), \( j = 1, \ldots, N \).

**References**
