Consistent Estimates in Functional Linear Models: 
Results and Simulations for LS approach 

Stime consistenti per modelli lineari funzionali: risultati e simulazioni per 
l’approccio LS

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1. Abstract

In some statistical situations the classical regression models may be inadequate: this 
is the case, for instance, when explanatory variables are points of a curve. Several 
examples can be found in different contexts as shown by Ramsay and Silverman (2005). 
Thus the functional linear models were introduced when the regressor is a process 
whose trajectories belong to the space of square integrable functions and the regression 
coefficient is given by an unknown time dependent function to be estimated following a 
sieve type least squares method which differs from the technique available in the literature. 
Furthermore, the mild assumptions introduced allow us to perform simulations in order 
to estimate the wide class of noncontinuous functions.

2. Introduction

In this paper we consider a functional linear model defined as follows

\[ Y = \int_0^\pi \theta_0(t)X(t)dt + \epsilon(X) \]  

where the regressor is a random element taking values into the separable Hilbert space 
\( L^2[0, \pi] \), \( \theta_0 \in L^2[0, \pi] \) is a given and unknown function over \( [0, \pi] \) and \( \epsilon(X) = \{ \epsilon(X) : x \in X \} \) denotes a family of probability distributions over \( \mathbb{R}^1 \) which is parametrized by 
the values \( x \in L^2[0, \pi] \) taken by the regressor \( X \). Our purpose consists in giving some 
theoretical results concerning the identifiability and the least squares estimates \( \hat{\theta}_n \) of the unknown function \( \theta_0 \). Furthermore a simulation study will be performed including the 
relevant case of a not continuous function \( \theta_0 \). The estimation technique we follow is 
consistently different from the approach usually adopted in the literature and based on 
spectral analysis of the empirical second moment operator of \( X \) (see Bosq, (1991) and 
Cardot et al. (1999)). In fact we use a sieve-type least squares estimation procedure. 
On the base of the i.i.d. observations \( \{(x_i, y_i) : i = 1, 2, \ldots, n \} \) the convex function is introduced

\[ L_n(\theta) = \frac{1}{n} \sum_{i=1}^n \left( y_i - \int_0^\pi \theta(t)x_i(t)dt \right)^2 \]
defined for all $\theta$ belonging to the finite dimensional subspace $Sp \left\{ \frac{\sin(jt)}{\sqrt{\pi/2}} : j = i, \ldots, m \right\}$ generated by the first $m_n$ elements of the orthonormal base and having its unique global minimizer $\hat{\theta}_n$. Under the assumptions below and when the number $n$ of observations tends to infinity, the sequence $\{L_n(\cdot)\}$ converges almost surely and uniformly over each fixed ball $\overline{S}_{\|\|}(0, r) \subset L^2[0, \pi]$ to the limit function $L(\theta) = E \left[ \left( Y - \int_0^\pi \theta(t)X(t)dt \right)^2 \right]$, $\forall \theta \in L^2[0, \pi]$.

3. Assumptions and main results

The assumptions below introduce very mild conditions under which the results described in this paper hold true. A direct comparison with the hypothesis adopted in the literature shows the gain we obtain in terms of generality.

A1 The regressor $X$, defining a measurable random element taking values into $L^2[0, \pi]$, satisfies the integrability condition $E(\|X\|^2) < \infty$.

A2 Denoting by $P_X$ the probability measure induced by $X$ over the Borel $\sigma$-field on $L^2[0, \pi]$, we have that $P_X(\alpha) < 1$ for any hyperplane $\alpha \subset L^2[0, \pi]$.

A3 Denoting by $\sigma^2(x) = Var(\epsilon(x))$ the error variance. We suppose that $\sigma^2(x)$ is a positive and measurable function satisfying the integrability condition $\int_{L^2[0, \pi]} \sigma^2(x) dP_X(x) < \infty$.

The above assumptions allow us to solve the identifiability problem for $\theta_0$ by the following

**Theorem** Under the assumptions A1, A2 and A3 the unknown parameter $\theta_0 \in L^2[0, \pi]$ is the unique global minimizer for the strictly convex function $L(\cdot)$.

Thus the strong consistency is proved for the LS estimates $\hat{\theta}_n$ of $\theta_0$ in the weak topology; i.e. the almost sure convergence is stated $\lim_{n \to \infty} \rho(\hat{\theta}_n, \theta_0) = 0$, where $\rho$ denotes the metric which defines the weak topology over each fixed closed ball $\overline{S}_{\|\|}(0, r)$.

The result is then complete, under the operative point of view, proving that a sequence $\{m_n\}$ can be always derived in such a way that $\sup_n \|\hat{\theta}_n\| < \infty$ for each assigned sequence of i.i.d. observations $(\bar{x}, \bar{y})$ belonging to a set of probability one.

**References**

