Permutation Tests for Umbrella Alternatives

Test di permutazione per alternative ad ombrello

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1. Introduction

There is a wide variety of stochastic ordering problems where \( K \) groups (typically ordered with respect to time) are observed along with a (continuous) response. For instance, consider the effect of a drug, which is typically increasing up to a certain point \( \hat{k} \), and then it decreases. The interest of the study may be on finding the change-point group, i.e. the group where an inversion of trend of the variable under study is observed. A change point is not merely a maximum (or a minimum) of the time-series function, but a further requirement is that the trend of the time–series is monotonically increasing before group \( \hat{k} \) and monotonically decreasing afterwards.

2. Testing for umbrella alternatives

A suitable solution can be provided within a conditional approach, i.e. by considering some suitable nonparametric combination of dependent tests for simple stochastic ordering problems. Let \( y_{ik} \) be the observed variable on the \( i^{th} \) subject from group \( k = 1, \ldots, K \). Let \( y_{ik} \) follow the additive model: \( y_{ik} = \mu + \delta_k + \epsilon_{ik} \), where \( \mu \) is the population mean, \( \delta_k \) is the effect of the variable under study at the \( k^{th} \) time – point, and \( \epsilon_{ik} \) are exchangeable errors with zero mean and finite variance \( \sigma^2 \). If we know the true time point \( \hat{k} \), then we could perform a statistical test to assess the alternative hypothesis:

\[ H_{\hat{k}0}: \delta_1 = \delta_2 = \ldots = \delta_{\hat{k}} = \delta_{\hat{k}+1} = \ldots = \delta_K \]

The alternative hypothesis entails the intersection of two partial alternative hypotheses:

\[ H_{\hat{k}1} = H_{\hat{k}0} \cap H_{\hat{k}1} \]

The related partial null hypotheses can be specified according to usual simple stochastic ordering problems:

\[ H_{\hat{k}0} \]

In order to perform a simple stochastic ordering test, i.e. testing for \( H_{\hat{k}1} \) against the alternative \( H_{\hat{k}1} \), a suitable test statistic is the following:

\[ T_{\hat{k}} = \sum_{j=1}^{\hat{k}} \sum_{i=1}^{\hat{k}} y_{j1} \ldots y_{\hat{k}} - y_{1\hat{k}+1} \ldots y_{j}, \]
where $\hat{y}_{j+1,...,k}$ is the sampling mean of the pooled groups $j+1,...,k$. Note that each element in the statistic (1) is significant versus $H_{ik}^\uparrow$ for large values. Similarly, we can define a test statistic to assess $H_{ik}^\downarrow$ by letting:

$$T_{k}^\downarrow = \sum_{j=k}^{K-1} \bar{y}_{j,...,k} - \bar{y}_{j+1,...,K}$$

(2)

The null distributions of (1) can be obtained by noting that $H_{ik}^\uparrow$ involve the exchangeability of the observations between the pooled vectors made of groups $(\hat{k}, \hat{k} + 1, ..., j)$ and $(j+1, ..., K)$. Thus, each argument in (1) is a two sample test statistic between groups $(\hat{k}, \hat{k} + 1, ..., j)$ and $(j+1, ..., K)$. In order to perform a global test statistic to assess $H_{ik}$, the nonparametric combination of dependent tests (Pesarin, 2001) can be applied. Let $\pi_{k}^\uparrow$ and $\pi_{k}^\downarrow$ be the p-values of respectively $T_{k}^\uparrow$ and $T_{k}^\downarrow$. Then a suitable global test can be defined as:

$$T_{k} = -2[\log(\pi_{k}^\uparrow) + \log(\pi_{k}^\downarrow)].$$

(3)

Clearly, the test statistic (3) assumes large values if both $\pi_{k}^\uparrow$ and $\pi_{k}^\downarrow$ are significant versus their related alternatives, therefore (3) allow to test for $H_{ik}$. Since the change-point $\hat{k}$ is usually unknown, we can find a possible significant change-point by repeating the above testing procedure for $\hat{k} = 1,...,K$. Let $\pi_{j}$ (j=1,...,K) be the p-values related to the test statistic (3) as if the $j^{th}$ group were the known change point group. A global test statistic to assess the presence of a significant change-point is given by $T_{G} = 1 - \min_{j} \pi_{j}$, which is also significant for large values. If the global test is significant, then the (true) change point group(s) is (are) the one(s) for which $\pi_{\hat{k}} = \min_{j} \pi_{j}$.

References
