A Bayesian Test for Normality Against Skewed Alternatives

Un test bayesiano di normalità contro alternative asimmetriche

Cinzia Carota
Dipartimento di Statistica e Matematika Applicata, Università di Torino
e-mail: carota@econ.unito.it

Keywords: divergence-based tests, locally best invariant tests, skew normal distribution

1. Introduction, parametrization of the alternatives and priors

The class of skew normal random variables, written $Y \sim SN(\mu, \sigma, \lambda)$, with density

$$f(y|\mu, \sigma, \lambda) = \frac{2}{\sigma} \phi\left(\frac{y-\mu}{\sigma}\right) \Phi\left(\frac{\lambda(y-\mu)}{\sigma}\right), \quad -\infty < y < \infty,$$

$-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \lambda < \infty$, where $\phi$ and $\Phi$ are the standard normal density function and distribution function respectively, has been introduced by Azzalini (1985).

We transform the skewness parameter $\lambda$ so that the range of the transformed parameter is $(-1, 1)$:

$$\delta = \frac{\lambda}{\sqrt{1 + \lambda^2}}.$$

Some possible interpretations of $\delta$ associated with the forms of genesis of the SN random variables can be found in Azzalini (2006). When $\delta = 0$ the $SN(\mu, \sigma, \delta)$ class of densities reduces to $N(\mu, \sigma)$, while otherwise the sign of $\delta$ gives the sign of the skewness. The location and scale parameters, $\mu$ and $\sigma$, will be treated as nuisance parameters and are therefore assigned a default prior, $\pi(\mu, \sigma) \propto 1/\sigma$; conversely for the parameter of interest, $\delta$, we adopt an independent prior very concentrated around the null value $\delta = 0$,

$$\pi(\delta|\alpha) = \frac{\Gamma(2\alpha)}{2^{2\alpha-1} \Gamma(\alpha)^2} (1 - \delta^2)^{\alpha - 1}, \quad \alpha \geq 1.$$

The peakness around the null value increases with the hyperparameter $\alpha$, since $Var(\delta) = 1/(\alpha + 1/2)$, and represents our prior degree of belief in the normal model. (This is equivalent to assuming $\lambda|\alpha \sim t(0, 1/(2\alpha); 2\alpha)$, where $t$ denotes a Student $t$ distribution with $2\alpha$ degree of freedom, zero mean and scale parameter equal to $1/(2\alpha)$.)

In the next section we provide a Jeffreys divergence-based test for the null hypothesis $H_0 : \delta = 0$ against the alternative $H_1 : \delta \neq 0$. A Taylor approximation of this test is interestingly related to the classical locally best invariant (LBI) test for $H_0 : \lambda = 0$ against $H_1 : \lambda > 0$ provided by Salvan (1986).

2. Test for normality in the class of $SN(\mu, \sigma, \delta)$ distributions

Given a random sample $y = y_1, ..., y_n$ from the $SN(\mu, \sigma, \delta)$ distribution, according to Carota (2005) the normal model can be rejected for large values of the Jeffreys divergence

\[ (*) \text{ Partially supported by MIUR, PRIN 2006, project: "Classi flessibili di distribuzioni ottenute mediante perturbazione della simmetria: aspetti probabilistici, statistici ed applicativi".} \]
between the prior and the posterior distributions of the skewness parameter,\[ \Delta = \frac{1}{2} \left\{ \int_{-1}^{1} \pi(\delta|a,y) \log \left( \frac{\pi(\delta|a,y)}{\pi(\delta|a)} \right) d\delta + \int_{-1}^{1} \pi(\delta|a) \log \left( \frac{\pi(\delta|a)}{\pi(\delta|a,y)} \right) d\delta \right\}. \]

By approximating \( \Delta \), through a Taylor series expansion of the logarithmic function around the null value \( \delta = 0 \), after some algebra, we obtain\[ \Delta_T = \frac{1}{2} \left[ E(\delta^3|y) - E(\delta^3) \right] l'''(\delta|y) \bigg|_{\delta=0}, \]

where \( l''' \) denotes the third derivative of the log of the integrated likelihood evaluated at \( \delta = 0 \). Observing that \( E(\delta^3|y) - E(\delta^3) < 2 \) and that at \( \delta = \lambda = 0 \) it turns out to be \( l'''(\delta|y) = l'''(\lambda|y) = I_S \),\[ I_S = \frac{2(\frac{4}{\pi} - 1)\sqrt{n}}{\sqrt{\frac{1}{2}}} \hat{\gamma}_1, \quad \hat{\gamma}_1 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^3}{n(s^2)^{\frac{3}{2}}}, \quad s^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n}, \]

where \( I_S \) is the classical LBI test statistic for normality in the \( SN(\mu,\sigma,\lambda), \lambda > 0 \), distribution class (Salvan, 1986) and \( \hat{\gamma}_1 \) is the Fisher statistic, we can conclude that two facts hold:
(i) \( I_S \) represents an upper bound for \( \Delta_T \) irrespectively of the prior assigned to the skewness parameter \( \delta \),\[ \Delta_T < I_S, \]

and (ii) for the given prior \( \pi(\delta|a) \)
\[ \Delta_T = E(\delta^3|y) \left( \frac{4}{\pi} - 1 \right) \sqrt{n} B\left(\frac{n-1}{2}, \frac{1}{2}\right) \hat{\gamma}_1, \]

where \( B(\cdot) \) denotes the Beta function. This fact assigns to the hyperparameter \( a \) a role in the calibration of \( \Delta_T \) parallel to that of the significance level in the calibration of a classical test statistic. (For a discussion of this point see Carota, 2007, section 4).

References