Asymmetric Multivariate Tail Dependence
La dipendenza di coda asimmetrica in ambito multivariato

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Keywords: copula function, financial returns, tail dependence

1. Introduction

The aim of the work is the analysis of the tail dependence in a multivariate context for financial returns. The approach consists in finding an appropriate ARCH-type model for univariate financial returns and in using a copula function to model the dependence of residuals. In particular, the class of so-called Archimedean copulas is really interesting because it allows for lower and/or upper tail dependence.

2. Copula functions and tail dependence

A copula function is used to describe the dependence structure among random variables, independently on the specifications of the marginal distributions. According to the theorem proposed by Sklar in 1959, given \( n \) continuous random variables \( X_1, X_2, \ldots, X_n \) with distribution functions \( F_1, F_2, \ldots, F_n \), the joint distribution function \( F(x_1, x_2, \ldots, x_n) \) can be written as

\[
F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).
\]

If we denote the marginal distribution function \( F_i = U_i \) and \( F_i(x_i) = u_i \), then \( U_i \) is uniform in \([0, 1]\) and

\[
C(u_1, \ldots, u_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(U_1 \leq u_1, \ldots, U_n \leq u_n).
\]

The most popular copula functions can be divided into two main classes: Elliptical and Archimedean. In a bivariate context, Archimedean copulas are particularly useful in measuring the dependence between two variables (cfr. Joe, 1997), such as the tail dependence, that is the amount of dependence among extreme values. Given two uniform random variables, \( U_1 \) and \( U_2 \) each defined in \([0, 1]\), the lower tail dependence \( \lambda_L \) is given by

\[
\lambda_L = \lim_{v \to 0^+} P(U_1 < v | U_2 < v),
\]

in terms of copula

\[
\lambda_L = \lim_{v \to 0^+} \frac{C(v, v)}{v},
\]

and the upper tail dependence is defined similarly.

Moving towards a multivariate perspective, we are interested in a multivariate tail dependence as well. In the 3-dimensional case, given three uniform random variables,
$U_1$, $U_2$ and $U_3$ each defined in $[0, 1]$, we can compute three (bivariate) tail dependence. Moreover, we can also compute three multivariate tail dependences, the first expressed as

$$
\lambda^{123}_L = \lim_{v \to 0^+} P(U_1 < v | U_2 < v, U_3 < v),
$$

in terms of copula

$$
\lambda^{123}_L = \lim_{v \to 0^+} \frac{C(v, v, v)}{C(1, v, v)}.
$$

Our empirical analysis has focused on daily returns of three stock indexes, DAX ($X_1$), MIB ($X_2$) and SP ($X_3$). After estimating separately a Threshold GARCH for each time series, we joined the marginals through several copula functions. In particular, the Clayton copula is able to capture a lower tail dependence (for financial returns the dependence among lower extreme values is effective for risk management). The general formulation of the Clayton copula function is

$$
C(u_1, \ldots, u_n) = \left[ \sum_{j=1}^{n} u_j^{-\theta_1} - (n - 1) \right]^{-1/\theta_1}.
$$

We estimated the bivariate and trivariate Clayton copula function. In the latter case we focused on an asymmetric version as well, $C(u_1, u_2, u_3) = C_1(C_2(u_1, u_2), u_3)$, characterized by an additional parameter. Table 1 contains the estimates of the parameter of the bivariate Clayton copula functions and of the (bivariate) lower tail dependences between the indexes. Table 2 contains the estimates of the parameters of the trivariate Clayton copula functions and of the lower tail dependence $\lambda^{312}_L$. The estimates are very different, remarking the importance of choosing a sufficiently general copula function to capture the concordance among extreme values.

### Table 1: Tail dependences using the bivariate Clayton copula

<table>
<thead>
<tr>
<th>Time-series</th>
<th>$\hat{\theta}_1$ (s.e.)</th>
<th>$\hat{\theta}_2$ (s.e.)</th>
<th>$\hat{\lambda}_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP - DAX</td>
<td>0.5539 (0.0386)</td>
<td>0.2861</td>
<td></td>
</tr>
<tr>
<td>SP - MIB</td>
<td>0.5443 (0.0389)</td>
<td>0.2799</td>
<td></td>
</tr>
<tr>
<td>DAX - MIB</td>
<td>1.2119 (0.0489)</td>
<td>0.5644</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: Tail dependences using the trivariate symmetric and asymmetric Clayton copula

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\theta}_1$ (s.e.)</th>
<th>$\hat{\theta}_2$ (s.e.)</th>
<th>$\hat{\lambda}^{312}_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm Clayton</td>
<td>0.6810 (0.0260)</td>
<td>-</td>
<td>0.5513</td>
</tr>
<tr>
<td>Asymm Clayton</td>
<td>0.5154 (0.0319)</td>
<td>1.1788 (0.0489)</td>
<td>0.3420</td>
</tr>
</tbody>
</table>

### References