GARCH models with spatial structure

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Riassunto: Questo lavoro presenta una breve rassegna su una nuova classe di modelli GARCH multivariati per la modellizzazione della volatilità di rendimenti azionari, chiamata Spatial Effects in ARCH, SEARCH. La dipendenza spaziale è associata alla classificazione delle imprese in settori industriali e in classi di capitalizzazione. Questa parametrizzazione risulta stimabile anche per ampie dimensioni sezionali, con un numero di parametri che è lineare nella dimensione sezionale, a differenza di modelli alternativi.

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1. Introduction

As it is well recognized in applied econometrics, unrestricted multivariate GARCH models lack parameter parsimony, at least for data-sets with a large cross-section dimension; see e.g. the review paper by Bauwens et al. (2006). In general multivariate GARCH specifications, the number of parameters — i.e. the model dimension — is in fact proportional to the fourth power of the number of assets, see Engle and Kroner (1995). This feature has limited the application of unrestricted GARCH models to systems of limited dimensions, i.e. typically with up to a handful of assets.

We consider a spatial approach in the specification of a GARCH models for stock returns. This model class is called SEARCH, short for Spatial Effects in ARCH; it was proposed and applied in Caporin and Paruolo (2005a, b) – abbreviated CPa and CPb. We provide here a summary of this specification, as well as of the empirical findings.

SEARCH models are designed for possibly large cross-sections of asset returns. Other feasible alternative specifications for large cross-sections include the constant conditional correlation (CC) model of Bollerlev (1990) and the orthogonal GARCH of Alexander (2001), OG. Unlike the CC and OG models, the SEARCH specification has number of parameters that is linear in the cross-section dimension, see CPa. Empirically, see CPb, it was found that the SEARCH specification favorably compares with alternatives.

2. A general class of GARCH models

Consider an \( n_x \times 1 \) dimensional vector time series \( \{x_t\}_{t \in \mathbb{N}} \) and the associated filtration \( \mathcal{I}_t := \sigma(x_{t-q}, q \geq 0) \). Let also \( x_t := (y_t' : z_t')' \) be partitioned into a \( n \times 1 \) subvector of
variables of interest $y_t$ and a $n_z \times 1$ subvector of other information variables $z_t$. We assume that $x_t$ has finite second moments conditional on $\mathcal{I}_{t-1}$. Indicate by $\mathbb{E}_{t-1} (\cdot) := \mathbb{E} (\cdot | \mathcal{I}_{t-1})$ the conditional expectation operator; let $\mu_t := \mathbb{E}_{t-1} (y_t)$ be the conditional mean of $y_t$, and let $\varepsilon_t := y_t - \mu_t$ is the $n \times 1$ vector of deviations from the conditional mean.

We consider a parametric model for the conditional mean $\mu_t$, taken for simplicity to be linear, $\mu_t = \mu_w \psi_t$, where $w_t := \left( x_{t-1}' : \cdots : x_{t-p_w}' \right)'$ is a $n_w \times 1$ vector. We assume that the inter-class dependence of $\varepsilon_t$ due to asset proximity can be summarized by

$$\varepsilon_t = S \varepsilon_t + \eta_t, \quad (1)$$

where $\eta_t$ is a $n \times 1$ vector of random variables with $\mathbb{E}_{t-1} (\eta_t) = 0$ and $\mathbb{V}_{t-1} (\eta_t) = \mathbb{E}_{t-1} (\eta_t \eta_t') =: \Gamma_t$. The matrix $I - S$ plays the role of the matrix of covariance eigenvectors in OG models, see Alexander (2001). When $S$ has spatial structure, (1) defines a spatial autoregression (SAR) process, see Cressie (1993) eq. (6.3.8) and reference therein. In CC models $S = 0$.

We indicate by $\Sigma_t$ the conditional variance covariance matrix of $\varepsilon_t$, $\Sigma_t := \mathbb{V}_{t-1} (\varepsilon_t) = \mathbb{E}_{t-1} (\varepsilon_t \varepsilon_t')$. A direct consequence of assumption (1) is that

$$\Sigma_t := \mathbb{V}_{t-1} (\varepsilon_t) = (I - S)^{-1} \Gamma_t (I - S')^{-1}.$$ 

provided $I - S$ is invertible, which we assume in the following. Similarly to the CC class, we assume that the errors $\eta_t$ in (1) have constant correlations, i.e. $\Gamma_t = D_t R D_t$ where $D_t$ is diagonal and $D_t^2 = \text{diag} (h_t) : = \text{diag} (h_{t1}, \ldots, h_{tn})$ with (possibly) time varying $h_{ti}$ and $R$ is a time-invariant correlation matrix of the standardized innovations $\psi_t := D_t^{-1} (I - S) \varepsilon_t = D_t^{-1} \eta_t$, $\mathbb{V}_{t-1} (\psi_t) = \mathbb{E}_{t-1} (\psi_t \psi_t') = R$.

We consider GARCH$(p_E, p_A)$ dynamics for $h_t := \text{vecd} (D_t^2)$, the vector of diagonal elements of $D_t^2$, driven by $e_t := (\eta_{1t}^2, \ldots, \eta_{nt}^2)'$

$$h_t = \zeta + E (L) h_{t-1} + A (L) e_{t-1}, \quad (2)$$

where by definitions above $h_t = \mathbb{E}_{t-1} (\varepsilon_t)$, so that $e_t - h_t$ is an innovation with respect to $\mathcal{I}_{t-1}$. This allows to calculate multi-step ahead predictions for the conditional variances using recursions as in standard multivariate GARCH models. Here $E (L) := \sum_{l=1}^{p_E} E_l L^{-1}$ and $A (L) := \sum_{l=1}^{p_A} A_l L^{-1}$, $E_l, A_l$ are $n \times n$ matrices, $\zeta$ is a $n$-dimensional vector.

The present class of GARCH models includes the CC and OG as special cases, as well as the SEARCH specification discussed in the following section.

3. Spatial covariance structures

The SEARCH class of models is characterized as the submodel of (1)-(2) with the property that the matrices $S$, $E_t$, $A_t$, $R$ (SEAR) which define the conditional heteroscedasticity (CH) features of the model, have spatial structure. Spatial proximity is defined on the basis of the following asset classification scheme.

Let $q$ be the cross-section index in $y_{qt}$. We assume that there exists a time-invariant classification of the assets by sector and size. In other words we assume that there exist mutually exclusive and exhaustive sets $C_{ij}$ of indices, such that $q \in C_{ij}$ indicates that asset number $q$ belongs to sector $i$ and capitalization size $j$, $i = 1, \ldots, k$, $j = 1, \ldots, \ell$. 
We define also the aggregated label sets for each sector $\mathcal{C}_i := \bigcup_{j=1}^k \mathcal{C}_{ij}$, for each capitalization class $\mathcal{C}_j := \bigcup_{i=1}^k \mathcal{C}_{ij}$ and for the whole market $\mathcal{C}_\cdot := \bigcup_{i=1}^k \bigcup_{j=1}^l \mathcal{C}_{ij} = \bigcup_{i=1}^k \mathcal{C}_i = \bigcup_{j=1}^l \mathcal{C}_j = \{1, \ldots, n\}$. The spatial covariation specification considered in this paper is based on such a classification.

In particular we define spatial weight matrices as follows. We let $W_{ij} := W(\mathcal{C}_{ij}) := (w_{ql})$, where $w_{ql} := (n_{ij} - 1)^{-1} 1(q \in \mathcal{C}_{ij}) (1(l \in \mathcal{C}_{ij}) - \delta_{ql})$ is the normalized spatial matrix that defines as neighbors of the $q$-th unit the units that belong to the same class $\mathcal{C}_{ij}$; here $n_{ij} := \#(\mathcal{C}_{ij})$ is the cardinality of the set $\mathcal{C}_{ij}$, $\delta_{ql}$ is Kronecker’s delta and $1(\cdot)$ is the indicator function. This definition is used for $i = \cdot, 1, \ldots, k$, $j = \cdot, 1, \ldots, l$, i.e. also for the aggregated classes. The matrix $S$ in (1) is specified as

$$S = (s, W_\cdot) + \left( \sum_{i=1}^k s_i W_{i.} \right) + \left( \sum_{j=1}^l s_j W_{.j} \right) + \left( \sum_{i=1}^k \sum_{j=1}^l s_{ij} W_{ij} \right),$$

where each term in parenthesis is interpreted as the effect of a factor with as many levels as elements in the sum. The first one is interpreted as a market factor, the second as a sector factor, the third as a size factor, the last as an interaction factor. The $s_{ij}$ coefficients in (3) are scalar parameters (or possibly diagonal matrices).

Also the GARCH dynamics is restricted to satisfy a spatial specification. The GARCH dynamics is assumed to reflect a possible spatial dependence within each class $\mathcal{C}_{ij}$. Without loss of generality, assume that assets are ordered as follows: the first $n_{11}$ assets belong to the class $\mathcal{C}_{11}$, the next $n_{12}$ belong to the class $\mathcal{C}_{12}$, where $n_{ij}$ indicates the number of assets in class $\mathcal{C}_{ij}$.

Consider the partition of $e_t$ into subvectors $e_t := (e_{11,t}, e_{12,t}, \ldots, e_{1k,t}, e_{21,t}, \ldots, e_{k,k,t})'$, where $e_{ij,t}$ is the subvector of $e_t$ corresponding to the class $\mathcal{C}_{ij}$. Partition also $\eta_t, \xi_t$ and $h_t$ conformably. We assume that the $n \times n$ matrices $E_t, A_t \in E(L) := \sum_{l=1}^p E_l L^{l-1}$ and $A(L) := \sum_{l=1}^p A_l L^{l-1}$ are block diagonal, where blocks are conformable with the partition of $e_t$. We indicate the blocks of $E_t$ and $A_t$ corresponding to $e_{ij,t-1}$ as $E_{ij,l}$ and $A_{ij,l}$, respectively; similarly we indicate the corresponding blocks of $E(L)$, $A(L)$ as $E_{ij}(L)$, $A_{ij}(L)$.

In order to define a spatial structure on the $E_{ij,l}$, $A_{ij,l}$ matrices, we assume that assets are ordered within each class $\mathcal{C}_{ij}$ according to a proximity criterion, e.g. by capitalization size. We assume that $E_{ij,l}$ and $A_{ij,l}$ have a spatial structure of the following form:

$$E_{ij,l} := \sum_{q=0}^{m_{E_{ij,l}}} \beta_{ij,l,q} W_{i,j,q}^*, \quad A_{ij,l} := \sum_{q=0}^{m_{A_{ij,l}}} \alpha_{ij,l,q} W_{i,j,q}^*,$$

where $\beta_{ij,l,q}$ and $\alpha_{ij,l,q}$ are scalars (or diagonal matrices) and $W_{i,j,q}^* := I_{n_{ij}}$. For $q > 0$ the matrices $W_{i,j,q}^* := \text{diag}(W_{i,j,q}^*)^{-1} W_{i,j,q}^*$ are $n_{ij} \times n_{ij}$ normalized spatial weight matrices; here $\tau_n$ is a $n$-vector of ones. Two possible choices for the spatial matrices $W_{i,j,q}^*$ are $W_{i,j,q}^* := U_{ij}^q$ or $W_{i,j,q}^* := U_{ij}^q + U_{ij}^q$ where

$$U_{ij} := \begin{pmatrix} 0 & 0 \\ I_{n_{ij}-1} & 0 \end{pmatrix}.$$

For scalar $\alpha_{ij,l,q}$ and $\beta_{ij,l,q}$ parameters, this choice of $W_{i,j,q}^*$ implies that $E_{ij,l}$ and $A_{ij,l}$ are Toeplitz matrices.
These specifications have the following interpretation: the spatial structure within each block relates each stock within \( e_{ij,t} \) with the preceding one in the list; this is the case for specification \( W_{ij,q}^0 := U_{ij}^q \), which implies an upper triangular system. Alternatively, for specification \( W_{ij,q}^0 := U_{ij}^q + U_{ij}^q \), each stock within \( e_{ij,t} \) is related to the one preceding and the one following it in the list, which implies a symmetric Toeplitz matrix for the scalar spatial coefficients.

The conditions for positive definiteness of the conditional variance matrix require the \( EA \) dynamics to deliver always positive definite conditional variances \( h_t \). A sufficient condition for this is that the \( \zeta, \alpha \) and \( \beta \) parameters are positive.

Also the correlation matrix \( R \) is assumed to be block diagonal with diagonal blocks \( R_{ij} \), where the subscripts \( ij \) refer to the class \( C_{ij} \). \( R_{ij} \) describes intra-class correlations within class \( C_{ij} \). We consider the following spatial specification:

\[
R_{ij} = I_{n_{ij}} + \sum_{q=1}^{m_{ij}} \rho_{ij,q} W_{ij,q}^* \tag{6}
\]

where \( W_{ij,q}^0 := U_{ij}^q + U_{ij}^q \), \( W_{ij,q}^* := \text{diag}(W_{ij,q}^0)^{-1}W_{ij,q}^0 \), and \( U_{ij} \) is defined in (5) above. A special case is given by the spatial autoregression of order one, where \( \rho_{ij,q} = \rho^q_{ij} \) for some scalar \( \rho_{ij} \). Another special case of interest is given by the restriction \( \rho_{ij,q} = \rho_{ij} \) for some scalar \( \rho_{ij} \) and all \( q \). This corresponds to the spatial classification of all the assets within class \( C_{ij} \) as first-order neighbors. The latter case is found to be empirically supported in CPb on a cross-section of 150 assets from the New York stock exchange.

A number of possible extensions and variations can be considered on the basic scheme proposed here, see CPa, who also discuss the relation of the SEARCH model with other GARCH specifications.

References


